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Ising Models on Hyperbolic Graphs

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We consider Ising models on a hyperbolic graph which, loosely speaking, is a discretization of the hyperbolic plane $\mathbf{H}^2$ in the same sense as $\mathbf{Z}^d$ is a discretization of $\mathbf{R}^d$. We prove that the models exhibit multiple phase transitions. Analogous results for Potts models can be obtained in the same way.

KEY WORDS: Ising/Potts models; Fortuin–Kasteleyn random cluster models; hyperbolic graphs.

1. INTRODUCTION

Ising models on the finite-dimensional lattices $\mathbf{Z}^d$ have been extensively studied over the past half century. It has been proved that phase transitions occur provided $d \geqslant 2$,[6, 10, 18] i.e., there exists a critical inverse temperature β_c such that when β (the inverse temperature) is above β_c, there is nonuniqueness of the Gibbs states. For ferromagnetic Ising models with zero external field, it is convenient to focus on the Gibbs states ν^+, ν^-, and ν^f obtained as the limit of finite-volume systems $\{\sigma_x : x \in \Lambda \subset \mathbf{Z}^d\}$ as $\Lambda \to \mathbf{Z}^d$ with respectively plus, minus, and free boundary conditions (b.c.). When β is below β_c, the states ν^+ and ν^- (which are necessarily extremal) are identical and hence the Gibbs state is unique. When β is above β_c, then $\nu^+ \neq \nu^-$; if ν^f is a mixture of ν^+ and ν^-, it must be the symmetric mixture $(\nu^+ + \nu^-)/2$. This decomposition of ν^f as a mixture of ν^+ and ν^- is expected to be true on $\mathbf{Z}^d$—at least, it has been proved for $d = 2$ (refs. 1 and 13; see also ref. 16) and for $d > 2$ at all large β,[9] and at all but countably many β's.[14] In summary, for Ising models on $\mathbf{Z}^d$, although it has not been completely proved, it is expected that:

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1. If $\beta < \beta_c$, then the magnetization $M = 0$ (so $\nu^+ = \nu^-$ and there is a unique Gibbs state), and the two-point function $\langle \sigma_x \sigma_y \rangle_f \to 0$ as x and y separate.

2. If $\beta > \beta_c$, then $M > 0$ and $\nu^f = (\nu^+ + \nu^-)/2$, so $\langle \sigma_x \sigma_y \rangle_f = \langle \sigma_x \sigma_y \rangle_\pm \geqslant M^2 > 0$ for all x and y.

In this paper, motivated by the works of refs. 4 and 17, we consider Ising models on a graph $\mathbf{G}$ (its definition follows immediately) which is *roughly isometric* to the hyperbolic plane $\mathbf{H}^2$ (see, e.g., ref. 5 for the definitions of rough isometry and $\mathbf{H}^2$), and hence is an ε-net of $\mathbf{H}^2$.[4] In other words, $\mathbf{G}$ is a discretization of $\mathbf{H}^2$ in the same sense as $\mathbf{Z}^d$ is a discretization of $\mathbf{R}^d$. Our definition of $\mathbf{G}$ given below is straightforward; one does not need to know the definition of rough isometry or $\mathbf{H}^2$ to understand the definition of $\mathbf{G}$. One reason that we state that $\mathbf{G}$ is a discretization of $\mathbf{H}^2$ is that if one wants to study a continuum Ising model on a hyperbolic plane, then our model can serve as an approximation of the continuum model. Ising models on graphs similar to $\mathbf{G}$ were studied by Series and Sinai,[19] who constructed uncountably many mutually singular Gibbs states which, they believed, are extremal. Other stochastic geometric models on manifolds with negative curvatures can be found in ref. 15 and references therein (the above mentioned H^2 has curvature -1). The graph $\mathbf{G}$ is defined as follows. Let $\mathbf{T}'_k$ be a homogeneous tree with degree k (i.e. each site of $\mathbf{T}'_k$ has exactly $k+1$ neighbors) and let O be the origin of $\mathbf{T}'_k$. Define $\mathbf{T}_k$ to be the "forwarding" tree obtained by deleting one of the $k+1$ edges emanating from O. Then $\mathbf{G}$ is the graph obtained by adding to $\mathbf{T}_k$ edges connecting equal-level sites of $\mathbf{T}_k$ (see Fig. 1).

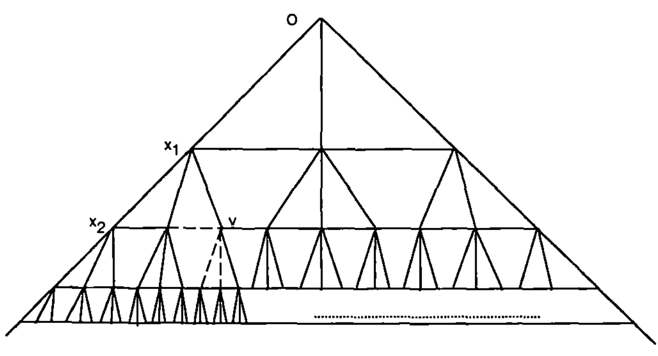

Fig. 1. $\mathbf{G}$ is obtained by adding to $\mathbf{T}_3$ horizontal bonds connecting equal-level sites of $\mathbf{T}_3$. The horizontal bond on the left side of v as well as the leftmost two "downward" bonds connecting v to its children are all closed (indicated by dashed bonds).

Independent percolation on **G** was studied by Benjamini,[4] who proved that the model exhibits multiple phase transitions in the sense that there exists none, a unique, and infinitely many infinite clusters, respectively, as the parameter p varies. It is known that there are strong analogies between nonuniqueness of infinite clusters in percolation and the occurrence of Gibbs states for Ising ferromagnets which are not mixtures of ν^+ and ν^-. Thus the results of Benjamini suggest that Ising models on **G** should also exhibit multiple phase transitions. The current paper substantiates this suggestion. Define $M \equiv M_O = \langle \sigma_O \rangle_+$ to be the magnetization at the origin, where $\langle \cdot \rangle_*$ denotes expectation with respect to the measure ν^* ($*$ is $+$ or $-$ or f). Note that although $M_x = \langle \sigma_x \rangle_+$ is dependent on x because the graph **G** is not translation invariant, it is easy to see by the FKG inequality that either M_x is identically equal to zero for all x in **G**, or $M_x > 0$ for all x in **G**. Moreover, from the structure of **G** it is not hard to see that $M_x \geqslant M_O$ for any x in **G**. We prove for sufficiently large k ($k \geqslant 8$ certainly suffices) that:

1. For small β, $M = 0$ (so $\nu^+ = \nu^-$ and there is a unique Gibbs state).
2. For intermediate β, $M > 0$ (so $\nu^+ \neq \nu^-$) and $\nu^f \neq (\nu^+ + \nu^-)/2$ (and there exists a sequence of pairs of sites $\{(x_i, y_i): i = 1, 2,...\}$ with $|x_i - y_i| \to \infty$ as $i \to \infty$ such that $\langle \sigma_{x_i} \sigma_{y_i} \rangle_f \to 0$ as $i \to \infty$, where $|x - y|$ denotes the distance between x and y).
3. For large $\beta, M > 0$ and $\nu^f = (\nu^+ + \nu^-)/2$ [and $\langle \sigma_x \sigma_y \rangle_f = \langle \sigma_x \sigma_y \rangle_\pm \geqslant M^2 > 0$ for any x and y in **G**].

We believe that the above results hold for any $k \geqslant 2$, although at present our proof does not work for small k; it requires k to be large enough to guarantee that the intermediate region is nonempty. One can improve k a little bit at the cost of a messier argument, but we do not see an argument which works up to $k = 2$. In Section 2 we state our results as several propositions. The statement is just for Ising models, but our proof applies to q-state Potts models with $q \geqslant 3$ as well, except that it requires larger k to guarantee the nonemptiness of the intermediate region. Proof of the propositions is given in Section 3.

2. PROPOSITIONS

The ferromagnetic Ising model on the graph **G** is described by the spin random variables $\{\sigma_x : x \in \mathbf{G}\}$. Each σ_x takes on the values ± 1. The interaction between the spins is described by the Hamiltonian

$$H = -\tfrac{1}{2}\beta \sum_{\{x, y\}} \sigma_x \sigma_y \tag{1}$$

in which the sum is over nearest neighbor bonds $\{x, y\}$, and $\beta \geqslant 0$ is the inverse temperature. Let ν^+ (respectively ν^- or ν^f) denote the Gibbs state associated with the Hamiltonian H with plus (respectively minus or free) b.c. Let $\langle \cdot \rangle_*$ denote the expectation with respect to ν^*, where $* = +, -$, or f. The (infinite-volume) quantities of primary interest to us are the magnetization at the origin

$$M \equiv M_O = \langle \sigma_O \rangle_+ \tag{2}$$

and the two-point function $\langle \sigma_x \sigma_y \rangle_*$ with $*$ b.c. From (7) below and the FKG inequality it is easy to see that either $M_x = \langle \sigma_x \rangle_+$ is identically zero for all x in **G** or else $M_x > 0$ for all x in **G**, and that $M_x \geqslant M$ for any x in **G**. Our results are stated in the following propositions.

Proposition 1. If $\beta < \ln[(k+2)/(k+1)]$, then $M = 0$ and consequently the Gibbs state is unique.

Proposition 2. If $\beta \geqslant \ln[(k+1)/(k-1)]$, then $M > 0$ and consequently there are more than one Gibbs state.

Proposition 3. If $\beta < (\ln k)/k$, then there exists a sequence of pairs of sites $\{(x_i, y_i)\colon x_i \in \mathbf{G}, y_i \in \mathbf{G},$ and $i = 1, 2,...\}$ (with $|x_i - y_i| \to \infty$ as $i \to \infty$) such that $\langle \sigma_{x_i} \sigma_{y_i} \rangle_f \to 0$ as $i \to \infty$. Consequently, if also $M > 0$, then

$$\nu^f \neq (\nu^+ + \nu^-)/2 \tag{3}$$

Proposition 4. If β is sufficiently large, then

$$\langle \sigma_x \sigma_y \rangle_f \geqslant \varepsilon > 0 \tag{4}$$

for any x and y in **G**, and

$$\nu^f = (\nu^+ + \nu^-)/2 \tag{5}$$

The propositions will be proved in the next section. We make several remarks here.

1. From Propositions 1 and 2 and the monotonicity of M as a function of β, there exists a critical point β_c in the interval $[\ln[(k+2)/(k+1)], \ln[(k+1)/(k-1)]]$ such that $M = 0$ if $\beta < \beta_c$ and $M > 0$ if $\beta > \beta_c$.

2. It is easy to check that $\ln[(k+1)/(k-1)] < (\ln k)/k$ when $k \geqslant 8$. So from Propositions 2 and 3, when $k \geqslant 8$ there exist $\beta'_c > \beta_c$ such that if $\beta_c < \beta < \beta'_c$, then $M > 0$ and $\nu^f \neq (\nu^+ + \nu^-)/2$. From Proposition 4, there exists $\beta''_c \geqslant \beta'_c$ such that if $\beta > \beta''_c$, then $M > 0$ and $\nu^f = (\nu^+ + \nu^-)/2$. It is natural to expect that $\beta'_c = \beta''_c$, but this has not been proved, since there is no such "monotonicity" which says that if $\nu^f = (\nu^+ + \nu^-)/2$ for some β_0,

then the same decomposition is also valid for any $\beta > \beta_0$. On the other hand, it would be very interesting if it happens that $\beta'_c < \beta''_c$, since this would mean that when β is between β'_c and β''_c, $\nu^f \neq (\nu^+ + \nu^-)/2$ and $\nu^f = (\nu^+ + \nu^-)/2$ alternately occur. But we believe this is not the case.

3. If one chooses to characterize the second phase transition by the behavior of the two-point function, then the situation becomes more clear. From Propositions 2–4 and the (weak) monotonicity of ν^f with respect to β, when $k \geqslant 8$ there exists the second critical point $\bar{\beta}_c > \beta_c$ such that if $\beta_c < \beta < \bar{\beta}_c$, then $M > 0$ and there are x_i and y_i in **G** with $\langle \sigma_{x_i} \sigma_{y_i} \rangle_f \to 0$ as $i \to \infty$, and if $\beta > \bar{\beta}_c$, then $M > 0$ and $\langle \sigma_x \sigma_y \rangle_f \geqslant \varepsilon > 0$ for all x and y in **G**.

3. PROOF OF PROPOSITIONS

The proof of the propositions is based on a result of Benjamin[4] and a new result about the connectivity function for independent percolation. These results are carried over to Ising models by use of the Fortuin–Kasteleyn (FK) representations of Ising models as dependent percolation models and Fortuin's comparison inequalities relating these dependent percolation models to independent ones. The FK random cluster models are described by probability measures on the configurations of bond variables, $n = \{n_b\}$, which take the value 1—meaning the bond $b = \{x, y\}$ is open, or 0—meaning b is closed. For a finite $\Lambda \subset \mathbf{G}$, the free b.c. measure $\mu^f_{\Lambda, q, p}$ has bond-configuration probabilities proportional to

$$q^{C(n)} p^{O(n)} (1-p)^{|\Lambda| - O(n)} \tag{6}$$

where $C(n)$ denotes the number of distinct clusters defined by the bond configurations n, $|\Lambda|$ is the number of bonds in Λ, and $O(n)$ is the number of open bonds in Λ. The "wired" b.c. measure $\mu^w_{\Lambda, q, p}$ is defined similarly, except $C(n)$ is determined by regarding all the sites in Λ^c, as well as those sites in Λ which are connected to Λ^c by an open path, as connected. For $q \geqslant 1$, infinite-volume measures $\mu^f_{q,p}$ and $\mu^w_{q,p}$ exist.[2, 7, 11] The $q = 1$ case is just the independent percolation model, where the probability measure is denoted by P_p. We will simply write μ^*_p for $\mu^*_{2,p}$, where $* = f$ or w. Let $p = 1 - e^{-\beta}$, then Ising models and FK random cluster models are related by the following identities:

$$M_x \equiv \langle \sigma_x \rangle_+ = \mu^w_p(x \leftrightarrow \infty) \tag{7}$$

$$\langle \sigma_x \sigma_y \rangle_f = \mu^f_p(x \leftrightarrow y) \tag{8}$$

$$\langle \sigma_x \sigma_y \rangle_\pm = \mu^w_p(x \leftrightarrow y, \text{ but } x \not\leftrightarrow \infty \text{ and } y \not\leftrightarrow \infty) + \mu^w_p(x \leftrightarrow \infty \text{ and } y \leftrightarrow \infty) \tag{9}$$

where $x \leftrightarrow y$ means that x and y are in the same open cluster, and $x \leftrightarrow \infty (x \not\leftrightarrow \infty)$ means that the cluster of x is infinite (finite). (See ref. 2 for more details.)

The FK random cluster model is related to the dependent percolation model by Fortuin's comparison inequalities stated in Lemma 1 below. For a proof of these inequalities see ref. 2, 7, or 11. For two probability measures μ and μ', we write $\mu \preccurlyeq \mu'$ if $\mu(A) \leqslant \mu'(A)$ for every increasing event A.

Lemma 1. For $q \geqslant 1$, let $\mu^*_{q,p}$ be a free or wired b.c. measure of the FK random cluster model in **G** and let P_p be the corresponding independent percolation measure. Then

$$\mu^*_{p,q} \preccurlyeq P_p$$

and

$$\mu^*_{p,q} \succcurlyeq P_{p'}$$

where $p' = p/(p + (1-p)q)$.

Although the lemma is stated for any $q \geqslant 1$, we only need the $q = 2$ case to prove the results for Ising models.

Proof of Proposition 1. From Lemma 1, we have that

$$M \equiv \mu^w_p(O \leftrightarrow \infty) \leqslant P_p(O \leftrightarrow \infty)$$

Because each site of **G** has at most $k+3$ neighbors, a standard elementary argument of independent percolation shows that $P_p(O \leftrightarrow \infty) = 0$ when $p < 1/(k+2)$, i.e., (recall $p = 1 - e^{-\beta}$) when $\beta < \ln[(k+2)/(k+1)]$. This proves the proposition. ∎

Proof of Proposition 2. Since **G** contains $\mathbf{T}_k$ as a subgraph and the critical point for independent percolation on $\mathbf{T}_k$ is $1/k$, the critical point for independent percolation on **G** is strictly less than $1/k$ by a result of Aizenman and Grimmett.[3] Applying Lemma 1 again, we have that for $p' \geqslant 1/k$,

$$M \equiv \mu^w_p(O \leftrightarrow \infty) \geqslant P_{p'}(O \leftrightarrow \infty) > 0$$

Recall that $p' = p/(p + (1-p)2)$ and $p = 1 - e^{-\beta}$, it is easy to see that $p' \geqslant 1/k$ is equivalent to $\beta \geqslant \ln[(k+1)/(k-1)]$. ∎

Proof of Proposition 3. Consider an independent *bond* percolation on **G** where each bond is open with probability p and closed with probability $1-p$. A site of **G** is called a boundary site if it is either on the left ray or on the right ray emanating from the origin O. A site which is not a boundary site is called an interior site. Let $y_n = O$ for $n = 1, 2,\ldots$, and let x_n be the nth-generation "child" of O along the left ray, as shown in Fig. 1. We first prove that when $p < 1 - 1/k^{1/k}$

$$P_p(x_n \leftrightarrow y_n) \to 0 \qquad \text{as} \quad n \to \infty \tag{10}$$

Then apply Lemma 1, we have that as $n \to \infty$

$$\langle \sigma_{x_n} \sigma_{y_n} \rangle_f = \mu_p^f(x_n \leftrightarrow y_n) \leqslant P_p(x_n \leftrightarrow y_n) \to 0 \tag{11}$$

when $p < 1 - 1/k^{1/k}$, or equivalently when $\beta < (\ln k)/k$.

To prove (10), consider the following independent *site* percolation on $\mathbf{T}_k$ (or more precisely in the interior of $\mathbf{T}_k$). An interior site v is called a *blocking site* if the "horizontal" bond on the left of v as well as the leftmost $k-1$ "downward" bonds from v to its children are all closed (see Fig. 1). So $P_p(v \text{ is a blocking site}) = (1-p)^k$. We say *$v$ percolates through blocking sites* if there exists an infinite sequence of sites $\{v_i : v_i \in \mathbf{T}_k, i = 0, 1, 2,\ldots\}$ such that $v_0 = v$, v_{i+1} is a child of v_i, and v_i is a blocking site for $i = 0, 1, 2,\ldots$. Then since the critical point for independent site percolation on $\mathbf{T}_k$ is $1/k$, we have that

$$D(p) \equiv P_p\,(v \text{ percolates through blocking sites}) > 0$$

when $(1-p)^k > 1/k$, or equivalently when $p < 1 - 1/k^{1/k}$. For $i = 1, 2,\ldots, n-1$, define B_i to be the event that the k "downward" bonds connecting x_i to its k children are all closed and at least one of the $k-1$ interior children of x_i percolates through blocking sites. Then when $p < 1 - 1/k^{1/k}$

$$P_p(B_i) = (1-p)^k[1 - (1 - D(p))^{k-1}] \geqslant (1-p)^k D(p) > 0$$

It is easy to see that if B_i occurs for some $i (1 \leqslant i \leqslant n-1)$, then there is no open connection from x_n to y_n. So the event "$x_n \leftrightarrow y_n$" is contained in $\bigcap_{i=1}^{n-1} B_i^c$. Noticing that the B_i are independent events, we have that

$$\begin{aligned} P_p(x_n \leftrightarrow y_n) &\leqslant P_p\left(\bigcap_{i=1}^{n-1} B_i^c\right) \\ &\leqslant [1 - (1-p)^k D(p)]^{n-1} \to 0 \qquad \text{as} \quad n \to \infty \end{aligned} \tag{12}$$

when $p < 1 - 1/k^{1/k}$. This proves (10) and hence (11).

Finally, if also $M > 0$, then on one hand $\langle \sigma_x \sigma_y \rangle_\pm \geqslant M_x M_y \geqslant M^2 > 0$ for any x and y in $\mathbf{G}$. On the other hand, by the just proved (11), $\langle \sigma_{x_n} \sigma_{y_n} \rangle_f \to 0$ as $n \to \infty$, so

$$\nu^f \neq (\nu^+ + \nu^-)/2$$

This completes the proof of Proposition 3. ∎

Proof of Proposition 4. Consider independent bond percolation on **G**. It is proved in ref. 4 that when p is sufficiently close to 1, then (i) there exists a unique infinite open cluster and (ii) there exist infinitely many disjoint open crossings from the left ray to the right ray of **G**. (The proof in ref. 4 is for site percolation, but the same argument applies to bond percolation as well.) From (i), we have that for p' sufficiently close to 1

$$\begin{aligned}\langle \sigma_x \sigma_y \rangle_f = \mu_p^f(x \leftrightarrow y) &\geqslant P_{p'}(x \leftrightarrow y) \\ &\geqslant P_{p'}(x \to \infty \text{ and } y \to \infty) \\ &\geqslant P_{p'}(x \to \infty)\, P_{p'}(y \to \infty) \\ &\geqslant P_{p'}^2(O \to \infty) > 0\end{aligned}$$

where the second inequality is due to uniqueness of the infinite open cluster and the third one is because of the FKG inequality. This proves the first part of the proposition.

From (i) and (ii), the unique infinite open cluster "cuts" **G** into (possibly infinitely many) "finite islands"; arguing as in the proof of Proposition 2.1.5 in ref. 17 we have

$$\nu^f = (\nu^+ + \nu^-)/2$$

The proof is completed. ▮

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